EQUATIONS FOR THE NON-STATIONARY TEMPERATURE FIELDS IN THIN SHELLS IN THE PRESENCE OF SOURCES OF HEAT

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If one neglects the inverse thermoelastic effect, which plays a very insignificant role, the problem of thermoelasticity falls into two parts, solved in sequence: the problem of determining the temperature field in the shell under given conditions of heat exchange and with given distribution of sources of heat; and the problem of determining the stress and deformation.

Let x^1 and x^2 be the curvilinear coordinates of the mean surface of the shell, and let $x^3 = z$ be the coordinate taken normal to the mean surface. Then, for the temperature T at a given point of the shell, the expression may be taken

$$T = T_0 (x^1, x^2) + z\Theta (x^1, x^2) \tag{1}$$

where T_0 is the temperature of the mean surface, and Θ is the temperature gradient in the direction of the normal to the mean surface. The expression (1) should be considered as a kind of analogue of the Kirchhoff-Love hypothesis in the theory of shells.

The problem consists of obtaining two differential equations connecting the functions T_0 and Θ. These same functions lead to differential equations for the thermoelastic deformation of the shell. Combining the equations, we shall obtain the entire system of equations of thermoelasticity for thin shells. The particular case of such equations for plates has appeared in the literature [1,2]. A generalization of the equations will be given below for the case of non-stationary fields in shells in the presence of sources of heat; incidentally, an inaccuracy in one of the equations of Melan and Parkus [1] is corrected.

We shall proceed from the variational principle for the problem of heat conduction. We consider the functional

$$I=\int_{t_0}^{t_1}\left(\int_V LdV+\int_S MdS\right)dt \tag{2}$$

$$L=\frac{c_p\rho}{2}\left(T^*\frac{\partial T}{\partial t}-T\frac{\partial T^*}{\partial t}\right)+\lambda\nabla_i T\,\nabla^i T^*-q\,(T+T^*),\quad M=k\,(TT^*-T_{\text{H}}T-T_{\text{H}}T^*)$$

Here c_p is the specific heat, ρ is the density of the material, λ is the coefficient of thermal conductivity, q is the density of thermal sources equal to the quantity of heat produced per unit time and unit volume of the body, k is the coefficient of thermal emission of the surface of the body and T_H is the temperature of the surrounding medium. Along with the temperature T, the temperature T^* is introduced in expressions (3) and (4), characteristic of the process taking place in the reverse direction (introduction of this process is necessary in order that the phenomenon as a whole should be conservative). The integrals on the right-hand side of (2) are taken over the entire volume V of the body and over the surface S; t_0 and t_1 are two arbitrarily chosen instants of time. $\nabla_i T$ and $\nabla^i T^*$ designate the covariant and contravariant derivatives with respect to the curvilinear coordinate x^i.

It is not difficult to recognize that the Ostrogradskii-Euler equation and the natural boundary condition for the variational problem $\delta I = 0$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{T}^*}+\nabla_i\frac{\partial L}{\partial(\nabla_i T^*)}-\frac{\partial L}{\partial T^*}=0,\qquad \frac{\partial L}{\partial(\nabla_i T^*)}n_i+\frac{\partial M}{\partial T^*}=0 \tag{3}$$

coincide with the equation of heat conduction and the condition of heat transfer on the surface S:

$$c_p\rho\frac{\partial T}{\partial t}-\lambda\nabla_i\nabla^i T=q,\qquad \lambda\nabla^i T n_i+k\,(T-T_{\text{H}})=0 \tag{4}$$

Here n_i is a vector parallel to the outward-drawn normal to the surface S.

We consider a thin shell of constant thickness h. The curvilinear coordinates of the mean surface will be designated by x^a (it is assumed in the following that the Greek indices will take on the values 1 and 2). The mean surface Ω is bounded by the contour Γ. The outer surface of the shell is designated by Ω_+ and the inner one by Ω_-; the corresponding temperatures of the surroundings are designated by T_+ and T_-. We assume that the thickness of the shell is sufficiently small in comparison with the radius of curvature that it is possible to set $\Omega_+\approx\Omega_-\approx\Omega$.

Replacing the integration over the volume V by an integration through the thickness of the shell from $-h/2$ to $h/2$ and an integration over the mean surface, and the integration over the surface S by an integration over Ω_+ and Ω_- and over the surface of the edge of the shell, we obtain in place of (2) the functional

$$I = \int_{t_0}^{t_1}\left(\int_{\Omega}\int_{-h/2}^{h/2} L d\Omega dz + \int_{\Omega} M_+ d\Omega + \int_{\Omega} M_- d\Omega + \int_{\Gamma}\int_{-h/2}^{h/2} M_\Gamma d\Gamma\, dz\right) dt \qquad (5)$$

Here M_+, M_- and M_Γ are expressions of the type of (4) for the surfaces Ω_+, Ω_- and for the edge surface. In conformity with the assumption (1) we set $T = T_0 + z\Theta$ and $T^* = T_0^* + z\Theta^*$. Carrying out the integration with respect to z, we easily obtain

$$\int_{-h/2}^{h/2} L dz = \frac{c_p \rho h}{2}\left(T_0^* \frac{\partial T_0}{\partial t} - T_0 \frac{\partial T_0^*}{\partial t}\right) + \frac{c_p \rho h^3}{12}\left(\Theta^* \frac{\partial \Theta}{\partial t} - \Theta \frac{\partial \Theta^*}{\partial t}\right) +$$
$$+ \lambda h (\nabla^\alpha T_0 \nabla_\alpha T_0^* + \Theta\Theta^*) + \frac{\lambda h^3}{12} \nabla^\alpha \Theta \nabla_\alpha \Theta^* - Q(T_0 + T_0^*) - Q z_0 (\Theta + \Theta^*)$$

In this expression

$$Q = \int_{-h/2}^{h/2} q dz, \quad z_0 = \frac{1}{Q}\int_{-h/2}^{h/2} q z dz$$

The former quantity is obviously the density of thermal sources per unit area of the mean surface; the parameter z_0 plays the role of the coordinate of the "center of gravity" of the sources. For M_+ and M_- we obtain the expressions

$$M_\pm = k\left[\left(T_0 \pm \frac{1}{2} h\Theta\right)\left(T_0^* \pm \frac{1}{2} h\Theta^*\right) - T_\pm\left(T_0 + T_0^* \pm \frac{h}{2}\Theta \pm \frac{h}{2}\Theta^*\right)\right]$$

Moreover

$$\int_{-h/2}^{h/2} M_\Gamma dz = kh\,[T_0 T_0^* - T_\Gamma (T_0 + T_0^*)] + \frac{kh^3}{12}[\Theta\Theta^* - \Theta_\Gamma (\Theta + \Theta^*)]$$

Here T_Γ and Θ_Γ are the mean temperature and temperature gradient on the edges.

The equations of Ostrogradskii-Euler for the variational problem $\delta I = 0$, where I is determined in conformity with (5), has the form

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial T_0^*} + \nabla_\alpha \frac{\partial L}{\partial(\nabla_\alpha T_0^*)} - \frac{\partial}{\partial T_0^*}(L + M_+ + M_-) = 0$$
$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \Theta^*} + \nabla_\alpha \frac{\partial L}{\partial(\nabla_\alpha \Theta^*)} - \frac{\partial}{\partial \Theta^*}(L + M_+ + M_-) = 0$$

From this we obtain the equations

$$\frac{\partial T_0}{\partial t} - \chi\triangle T_0 + \frac{2kT_0}{c_p\rho h} = \frac{Q}{c_p\rho h} + \frac{k}{c_p\rho h}(T_+ + T_-) \tag{6}$$

$$\frac{\partial\Theta}{\partial t} - \chi\triangle\Theta + \left(\frac{12\chi}{h^2} + \frac{6k}{c_p\rho h}\right)\Theta = \frac{12Qz_0}{c_p\rho h^3} + \frac{6k(T_+ - T_-)}{c_p\rho h^2} \tag{7}$$

where $\Delta = \nabla^\alpha\Delta_\alpha$, and $\chi = \lambda/c_p\rho$ is the coefficient of heat conduction.

The natural boundary conditions

$$\frac{\partial L}{\partial(\nabla_\alpha T_0^*)}n_\alpha + \frac{\partial M_\Gamma}{\partial T_0^*} = 0, \qquad \frac{\partial L}{\partial(\nabla_\alpha\Theta^*)}n_\alpha + \frac{\partial M_\Gamma}{\partial\Theta^*} = 0$$

give the conditions of heat exchange on the edges of the shell:

$$\lambda\nabla^\alpha T_0 n_\alpha + k(T_0 - T_\Gamma) = 0, \qquad \lambda\nabla^\alpha\Theta n_\alpha + k(\Theta - \Theta_\Gamma) = 0 \tag{8}$$

We notice that the problems of determining the temperature on the mean surface T_0 and the gradient Θ turn out to be completely independent for a thin shell.

The special case of Equation (6) for a stationary field in a plate in the absence of heat sources is reported in [1,2] where an averaging of the equation of heat conduction is used. An equation analogous to Equation (7) was derived in [1] in the form (according to our nomenclature)

$$-\chi\triangle\Theta + \frac{6k}{c_p\rho h}\Theta = \frac{6k(T_+ - T_-)}{c_p\rho h^2} \tag{9}$$

Thus, it does not agree with Equation (7). It is easy to see that Equation (9) is in error. In the case $\Delta T_0 = \Delta\Theta = 0$, the temperature of the free surface is equal to the temperature of the surrounding medium, and this is in contradiction to the conditions of heat exchange on the surface. Equation (7) is free from this inaccuracy.

To go over to a concrete curvilinear system of coordinates x_1 and x_2 with Lamé coefficients H_1 and H_2, it is sufficient to replace the Laplace operator according to the well-known formula

$$\triangle = \frac{1}{H_1H_2}\left(\frac{\partial}{\partial x_1}\frac{H_2}{H_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\frac{H_1}{H_2}\frac{\partial}{\partial x_2}\right)$$

For example, for a circular conical shell with vertex angle β and for a choice of coordinates the distance x measured from the vertex along the generating line, and the polar angle ϕ, we obtain

$$\frac{\partial T_0}{\partial t} - \chi\left(\frac{1}{x}\frac{\partial T_0}{\partial x} + \frac{\partial^2 T_0}{\partial x^2} + \frac{1}{x^2\sin^2\beta}\frac{\partial^2 T_0}{\partial\varphi^2}\right) + \frac{2kT_0}{c_p\rho h} = \frac{Q}{c_p\rho h} + \frac{k}{c_p\rho h}(T_+ + T_-) \tag{10}$$

$$\frac{\partial\Theta}{\partial t} - \chi\left(\frac{1}{x}\frac{\partial\Theta}{\partial x} + \frac{\partial^2\Theta}{\partial x^2} + \frac{1}{x^2\sin^2\beta}\frac{\partial^2\Theta}{\partial\varphi^2}\right) + \left(\frac{12\chi}{h^2} + \frac{6k}{c_p\rho h}\right)\Theta = \frac{12Qz_0}{c_p\rho h^3} + \frac{6k(T_+ - T_-)}{c_p\rho h^2}$$

The conditions (8) on the end of the shell $x = x_0$ have the form

$$\frac{\partial T_0}{\partial x} + k(T_0 - T_\Gamma) = 0, \qquad \frac{\partial \Theta}{\partial x} + k(\Theta - \Theta_\Gamma) = 0$$

Equations for temperature fields should be considered together with equations which describe the thermoelastic deformation in shells. Thus, for a shallow shell with thickness h, modulus of elasticity E, Poisson's ratio μ and thermal coefficient of expansion α in the case of deflections of the same order as the thickness, we obtain the system of equations

$$\frac{Eh^3}{12(1-\mu^2)}[\triangle\triangle w + \alpha(1+\mu)\triangle\Theta] = \tag{11}$$

$$= \frac{\partial^2\Phi}{\partial x_2^2}\left(k_1 + \frac{\partial^2 w}{\partial x_1^2}\right) + \frac{\partial^2\Phi}{\partial x_1^2}\left(k_2 + \frac{\partial^2 w}{\partial x_2^2}\right) - 2\frac{\partial^2\Phi}{\partial x_1\partial x_2}\frac{\partial^2 w}{\partial x_1 \partial x_2} + p(x_1, x_2, t) - \rho h \frac{\partial^2 w}{\partial t^2}$$

$$\frac{1}{Eh}\triangle\triangle\Phi + \alpha\triangle T_0 = \left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)^2 - \frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} - k_1\frac{\partial^2 w}{\partial x_2^2} - k_2\frac{\partial^2 w}{\partial x_1^2} \tag{12}$$

Here $w(x_1, x_2, t)$ is the normal deflection function, $\Phi(x_1, x_2, t)$ is the tangential force function, k_1 and k_2 are the principal curvatures of the mean surface (it is assumed that the lines of curvature coincide with the coordinate lines x_2 = constant and x_1 = constant), and $p(x_1, x_2, t)$ is the external normal loading per unit mean surface. Equation (11) includes an inertial term corresponding to normal displacements. Thus, the system is suitable for describing oscillations occurring with frequencies of the order of the natural frequencies of the normal oscillations, but small in comparison with the natural frequencies of the tangential oscillations. If, in consideration of non-stationary problems, the operational method is used, it is convenient to deal with the functions T_0 and Θ represented here rather than to go over to the original [3]. In this case, simultaneous consideration of Equations (6) and (7) with equations of the type of (11) and (12) is much more convenient.

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